Investigation of the Transverse mode of a laser with variation of the Hermite and Laguerre polynomial parameters.

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Abstract

This paper aims at explaining the operation of Hermite and Laguerre polynomials on the Gaussian beam in the case of higher order laser modes. The parameters of Hermite polynomial explain the Hermite–Gaussian laser beam in the Cartesian symmetry and that of Laguerre polynomial explains Laguerre–Gaussian laser beam in the cylindrical symmetry. In this paper it is shown that the Gaussian beam changes its shape as the parameters of the polynomials are changed and the change is explained. There is derivation of the two types of beam mode equations but how the beams develop as they propagate is illustrated in this paper. It gives the calculation of the energy and intensity of the laser just viewing the type of mode on the screen.

1. Introduction

Laser beam propagation is derived from the Helmholtz equation within the paraxial approximation [1-3]. The discussion of Gaussian and higher-order laser beams is a standard textbook topic in modern optics and laser physics courses [1-7]. Using the plane-wave representation of the fundamental Gaussian mode as a seed function, all higher-order beam modes can be derived by acting with differential operators [8]. The energy distribution can be calculated easily if the type of mode is known. By winding the fiber to a coil [9] with different radius, high-order modes can be used in multimode fiber. In the recent applications Laguerre–Gaussian (LG) and Hermite–Gaussian (HG) laser beams have interests in the field of optical trapping [10-15]. Higher order modes are also used in the experiments concerned with mode locking, optical resonators [16], optical pulse modeling [17], Z-scan [18, 19], multimode fiber [9], LCD [20] and many other fields of optics. In this paper the higher order modes: Hermite–Gaussian and Laguerre–Gaussian laser beams are shown schematically and explained how the two polynomials work on the Gaussian beam of lasers. The spot size of laser beam is also explained graphically. The spot size increases the radius of the spot without increasing the intensity. The parameters of the polynomials have significant effect on the Gaussian beam. The beam equations found in the textbooks explains the practical higher order laser beams. In this paper it is shown how the modes on the screen can be explained elaborately.

2. Theory

2.1. Hermite – Gaussian Laser Modes:

Three beam parameters such as a phase factor, a radius of curvature and a spot size are usually used to describe a Hermite Gaussian laser beam [24, 25]. Higher-order Hermite Gaussian beams always form a pattern of spots or a line of spots rather than a single spot of light [26]. Helmholtz equation gives us a function which is a product of two one dimensional functions [21] due to separation of variables for two dimensional rectangular coordinates. This can be explained by introducing the Hermite polynomial. Due to the Hermite–Gaussian laser beam the electric field distribution in the $z = 0$ plane is given by,

$$E_{m,n}(x,y) = E_0 H_m(\sqrt{\frac{x^2 + y^2}{w_0^2}}) H_n(\sqrt{\frac{x^2 + y^2}{w_0^2}}) \exp\left(-\frac{x^2 + y^2}{w_0^2}\right) \tag{1}$$

where, $H_m$ and $H_n$ are the Hermite polynomials and $m$ and $n$ are the transverse mode numbers. A Hermite–Gaussian beam field is expressed as a superposition of multiple fields at complex-source points [22]. The Hermite–Gaussian beam field can be expressed as the superposition of multiple fields at complex-source points in the paraxial region [23]. The beams with larger $m$ and larger wavelength $\theta$ have greater spreading in free space [27]. Experiments with soliton like steerable bright-soliton Y-junctions [28] is performed with second
order Hermite–Gaussian modes. High-quality beam can be formed from a pure high-order Hermite–Gaussian mode [29], a new method for performing scanning differential optical microscopy is based on the use of a higher-order Gauss–Hermite mode of a laser [30] and many other fields uses the Hermite–Gaussian laser modes. Various types of diffraction is experimented for Hermite–Gaussian laser modes [31–34].

2.2. Laguerre–Gaussian Laser Modes:
The above case in the cylindrical coordinate can be explained by the Laguerre polynomial $L_p^l$ which has two parts, $p$ is the radial part and $l$ is the angular part [35, 36]. Laguerre Gaussian modes can be produced inside the laser cavity [37], or by subsequent conversion from Hermite-Gaussian beams [38, 39]. The electric field distribution in this case is given by [21],

$$|\mathbf{E}(r, \phi)| = \mathbf{E}_0 \left( |\mathbf{E}_0| \right)^\frac{1}{2} L_p^l \left( \frac{r}{\sigma} \right) e^{-r^2/\sigma^2} e^{i\phi}$$  \hspace{1cm} (2)

Computer-generated hologram [40] is used for the production of high-order multiringed Laguerre–Gaussian modes.

3. Calculations

Graphical Representation of the two types of Laser Modes has been obtained.

3.1. Spreading of laser beam with spot – size ($w$):

Figure 1.1 and 1.2 shows the spreading of the size of the beam spot on the screen with spot – size variation ($w$). It is given for both Hermite – Gaussian and Laguerre – Gaussian laser modes. In both the cases the result is the same as the parameters are kept zero in both the cases.

3.2. Change in the shape of laser modes with the parameters of the polynomials ($m, n, p$ and $l$):

From figure 2.1–2.3 it is clear that the parameters $m$ and $n$ individually increases the number of spots in a line along only one axis. If $m$ is along the x axis then $n$ becomes along the y axis, since energy distribution is along the z axis. When both parameters act on the Gaussian laser beams, then the two lines of spots superimpose and a rectangular shape results in (figure 3.1–3.3). When the spots make a square then $m$ and $n$ are equal. The parameters of Laguerre polynomials also exhibit the cylindrical symmetry, $p$ is known as the radial mode number and $l$ is the angular mode number. When $p$ increases, the number of modes in the shape of rings increases and becomes equal to the value of $p$ (figure 4.1–4.3). The $360^\circ$ of the plane is divided into $2l$ equal angles and each angle contains an individual spot (figure 4.1–4.3). When $p$ and $l$ both operate on the Gaussian beam, the total number of spots is equal to the number of spots due to $l$ multiplied by the value $p + 1$ (Figure 5.1–5.3).

4. Discussion

This study gives a clear view about the two polynomials and how they describe the two types of transverse modes of the laser. Text books gave the two equations of energy distribution but how the two polynomials used in these two cases explain the two physical cases of laser transverse mode is not thoroughly given. To understand the effect of these two polynomials the three dimensional graphs are used here. These figures clearly give us the physical conditions from which we understand what is happening to the transverse modes in the cavity, when a definite type of spot is viewed on the screen. Also we can calculate the energy. These phenomena will be useful in the experiments with modes of laser which is a very important part of laser study. After developing the program written in Mathematica, changing the parameters it was tested, how the figures develop and change.

First the action of parameter $w$ which is called the “spot – size” was treated. As given by the theory, $w$ is the distance from the axis z, the changing value of $w$ changes the spot size that is the width of the beam in the xy plane (figures. 1.1 and 1.2). Keeping $w = .3$ we get a Gaussian shape as m and n were kept zero (figure 1.1(a)). When $w = .5$ was taken the spot took larger range in the xy plane (figure 1.1(b)). So, increasing the value of $w$ makes the beam fat that is the spot size is larger for large $w$ and it is shown from the top of the 3D plotted figure 1.2 which gives us the spots on the screen. Since the spot size experiment gave successful observation then $w$ was kept constant in the further observations.

Then for the transverse mode numbers $m$ and $n$ we found figures 2.1–2.3. Figure 2.1a shows that for $m = 6$ there are six spots along a line; it is also given as the top view of the 3D plot for both energy (figure 2.2.1.) and intensity distribution (figure 2.3.1.) for $m = 1$ and 2 respectively. With these figures it is clear that with the increasing value of $m$ the no. of spots or Gaussian beams increase along the one axis as the polynomial $H_m$.
operates with the variable $x$. A similar result is observed for $n = 5$, where the number of spots increase with increasing value of $n$ (figure 2.1b). There are some other examples for $n = 3$ (figures 2.2.2.a and 2.3.2.a) and 4 (figures 2.2.2.b and 2.3.2.b) in both the case of energy and intensity distribution. It was observed and point out that for all values of $m$ and $n$ there was one extra spot in each case. The spots at the end are bigger. Another important observation which needs to be pointed out that with increasing $m$ and $n$ the height of the beam in the $z$ axis increases exponentially. This means the intensity is greater when the number of modes is greater.

The next observation was the combined effect of $m$ and $n$. Combination of the shape of the spot takes a rectangular shape as seen from the top. As both of the transverse mode numbers $m$ and $n$ creates line of spots or beams along different axes as is evident in figures 3.1-3.3. The effect takes the shape of a square if the values are the same (figure 3.1b). Other examples of this case is figure 3.2.1 where $m = n = 1$ and figure 3.3.1 where $m = n = 2$. Rectangle shapes are created in figures 3.1a, 3.2.2 and 3.3.2.

The radial part of the Laguerre polynomial expands the spot in circular spots called ring like modes keeping a central spot (figures 4.1, 4.3.1 and 4.4.1). In figure 4.1 there are two examples for $p = 2$ and 4. The adjacent circle of the beams is on the negative side of plane in the case of energy distribution (figure 4.3.1) but for intensity distribution there is no negative side as it is the square of energy density (figure 4.4.1). This makes the central beam thinner and higher as the value of $p$ increases. The number of rings around the central spot is equal to the value of $p$.

The angular part $l$ makes a different impact on the Gaussian beam (figure 4.2, 4.3.2 and 4.4.2). The value of $l$ divides the $xy$ plane into two equal parts. When $l = 2$ the plane is divided into two parts twice (figure 4.2a) and when $l = 3$ the plane is divided into two parts thrice (figure 4.2b). There is a spot in each part. In the case of energy distribution, for even value of $l$ there are same numbers of Gaussian beam in the negative and positive side of energy axis (figure 4.3.2a) but for odd values there is an extra pair of beam in positive or negative direction (figure 4.3.2b). In the case of intensity distribution all the constructive beams form in the positive side as it should be (figure 4.4.2).

The combined effect gives number of spots equal to the multiplied value $2l(p + 1)$ as shown is figures 5.1-5.3.

In the case of cartesian coordinates, the transverse mode of the laser beam is described by the two Hermite polynomials with two transverse mode numbers. If we get interference patterns like figures 2.1-3.3.2 then we can use the Hermite polynomials to calculate the intensity. In case of cylindrical symmetry the transverse mode of the laser is described by the Laguerre polynomial, as given by figures 4.1-5.3.2.

So the three dimensional graphical representation of modes given by the two polynomials in the cartesian and cylindrical coordinates shows precisely how the transverse modes of the laser really work.

The intensity pattern in any experiment, gives the idea of the geometry of the resonator used and the laser intensity. Then form the intensity pattern on the screen it can be determined what the characteristics of the laser are. So, this study will be helpful in the further study of laser with laser modes and experiments concerned with laser modes.

**Figure 1.1.** Spot size for both Hermite and Laguerre – Gaussian Laser modes, when the parameters of the polynomials are zero or not operating. Side view:- a. $w = .3$; b. $w = .5$;

**Figure 1.2.** Top view :- a. $w = .1$; b. $w = .3$; c. $w = .5$; d. $w = .7$; e. $w = .9$;
Figure 2.1. Operation of the Hermite polynomials on the Gaussian beam individually. Side view:
- a. m = 6, n = 0;
- b. m = 0, n = 5;

Figure 2.2. Top view for energy distribution:
- a. m = 1, n = 0;
- b. m = 2, n = 0;

Figure 2.2.2. Top view for energy distribution:
- a. m = 0, n = 3;
- b. m = 0, n = 4;

Figure 2.3. Top view for intensity distribution:
- a. m = 1, n = 0;
- b. m = 2, n = 0;

Figure 2.3.2. Top view for intensity distribution:
- a. m = 0, n = 3;
- b. m = 0, n = 4;

Figure 3.1. Operation of both the parameters of the Hermite polynomial on the Gaussian beam. Side view:
- a. m = 2, n = 3;
- b. m = 6, n = 6;

Figure 3.2.1. Top view for energy distribution:
- a. m = 1, n = 1;
- b. m = 2, n = 2;

Figure 3.2.2. Top view for energy distribution:
- a. m = 2, n = 1;
- b. m = 1, n = 2;
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Figure 3.3.1. Top view for intensity distribution: a. m = 1, n=1; b. m = 2, n =2;

Figure 3.3.2 Top view for intensity distribution: a. m = 2, n = 1; b. m = 1, n = 2;

Figure 4.1. Individual operation of the Laguerre polynomials on the Gaussian beam. Side view: a. p = 2, l = 0; b. p = 4, l = 0;

Figure 4.2. Individual operation of the Laguerre polynomials on the Gaussian beam. Side view: a. p = 0; l = 2; b. p = 0, l = 3;

Figure 4.3.1. Top view for energy distribution: a. p = 1; b. p = 2;

Figure 4.3.2. Top view for energy distribution: a. l = 2; b. l = 3;

Figure 4.4.1. Top view for intensity distribution: a. p = 1; b. p = 2;

Figure 4.4.2. Top view for intensity distribution: a. l = 2; b. l = 3;
Figure 5.1. Operation of both the parameters of the Laguerre polynomial on the Gaussian beam. Side view: a. \( p = 1, l = 1 \); b. \( p = 1, l = 2 \);

Figure 5.2. Top view for energy distribution: a. \( p = 1, l = 1 \); b. \( p = 2, l = 2 \);

Figure 5.3. Top view for intensity distribution: a. \( p = 1, l = 1 \); b. \( p = 2, l = 1 \);

5. Conclusion

So from the above study it is found that, in the case of cartesian coordinates, the transverse mode of the laser beam is described by the two Hermite polynomials with two transverse mode numbers. In case of cylindrical symmetry the transverse mode of the laser is described by the Laguerre polynomial as explained. So the three dimensional graphical representation of modes given by the two polynomials, in the Cartesian and Cylindrical co-ordinates, shows precisely how the transverse modes of the laser really work.

As transverse modes are obvious in laser and since we cannot have plane waves in reality, it is of interest to investigate these modes in both the Cartesian and cylindrical coordinates.

The intensity pattern in any experiment, gives the idea of the geometry of the resonator used and the laser intensity. So, this study will be helpful in the further study of laser modes and experiments concerned with laser modes.

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